Notes on Existence of Flips

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On the existence of flips pt 3.

Reminder: Goal is to show that $\mathbb{R}$-MMP in dim $n-1 \Rightarrow$ Existence ~~of~~ flips in dim n

Essentially reduces MMP to termination

How to prove: • First reduce to PL flips (Shokurov)

- Define restricted algebra, R_S, on coefficient 1 part ~~of~~ the boundary, S.
- Under seasonable hypotheses, Restricted algebra is f.g. $\iff$ algebra upstairs is f.g. (iff the flip exists)
- Restricted algebra lives on S, dimension $n-1$ so we can use MMP

$n-1$ so we can use ...

by induction hypothesis.

- Construct model $T \to S$ and algebra $\mathcal{R}$ on T which is f.g. iff $\mathcal{R}_S$ is
- Show $\mathcal{R}$ is f.g. using

Theorem 8.3: Let Y be smooth, $\pi: Y \to Z$ projective, Z affine, normal. If $\mathcal{R}$ <u>saturated</u> <u>semiample</u> geometric algebra on Y then $\mathcal{R}$ is finitely generated.

Let's define these terms then prove the theorem.

Defn: $\pi: Y \to Z$ projective morphism of normal varieties, Z affine.

$\mathcal{R}$ is geometric algebra w/ additive sequence M_i of mobile divisors and characteristic sequence and characteristic sequence

$D_i \left(:= \frac{M_i}{i}\right)$

R is saturated if $\exists$ $\mathbb{Q}$-divisor F s.t. (1) $\lceil F \rceil \geq 0$ (all coeffs > -1)

(2) $\forall i, j \in \mathbb{Z}$, $\mathrm{Mob}\left(\lceil jD_i + F \rceil\right) \leq M_j$

Not a very intuitive definition but it's a sort of boundedness condition for the $M_\bullet$.

Defn: A geometric algebra R w/ additive sequence $B_\bullet$ is semiample if the characteristic sequence has a semiample limit.

We have seen that if a semiample algebra R, w/ characteristic sequence $D_\bullet$, satisfies $D_K = \lim_{i \to \infty} D_i$

$\mathcal{D}$. satisfies $D_k = \lim_{i\to\infty} D_i$ for some k, then $\mathcal{R}$ is finitely generated, so we will show that the saturated property forces this to happen.

To prove Theorem 8.3

Need Lemma 8.1, Diophantine approximation:

projective map

affine, normal

Y smooth, $\pi : Y \to Z$

D semiample $\mathbb{R}$-divisor on Y, and $\varepsilon > 0$ rational.

$\exists$ Integral divisor M and $m \in \mathbb{Z}_{>0}$ s.t.

- $\operatorname{Supp} M \subseteq \operatorname{Supp} D$
- M is bpf
- $\|mD - M\| < \varepsilon$
- If $mD \geq M$ then $mD = M$

This is clear if D is a $\mathbb{Q}$-divisor by definition of semiample.

The proof in the paper isn't quite right. Shokurov Prelimiting flips has a good proof which is long but pretty readable.

Proof of Theorem 8.3:

$\mathbb{R}$ saturated w.r.t. divisor F semiample, characteristic sequence $D_\bullet$ $\varinjlim D_\bullet =: D$.

Let G be $\mathrm{Supp}(D)$. because $\lceil F \rceil \geq 0$ we can choose $\varepsilon > 0$ s.t. $\lceil F - \varepsilon G \rceil \geq 0$.

By diophantine approximation $\exists m \in \mathbb{Z}_{>0}$; M mobile (bpf really)

- $\|mD - M\| < \varepsilon$
- If $mD \geq M$ then $mD = M$

Then $mD + f = M + (mD - M) + f$
$\geq M + f - \varepsilon G$

because $\mathrm{Supp}(mD - M) \subseteq G$ and all coeffts of $mD - M$ are less in absolute value than ε by definition of sup norm.

So then $\mathrm{Mob}(\ulcorner mD + f \urcorner) \geq M$ because $\ulcorner mD + f \urcorner \geq M$

Then by definition of saturation,

$$\mathrm{Mob}(\ulcorner mD_i + f \urcorner) \leq M_m = mD_m$$

as $i \to \infty$ get

Convexity property

$$M \leq \mathrm{Mob}(\ulcorner mD + f \urcorner) \leq mD_m \leq mD$$

Thus $M = mD$

but more importantly $mD_m = mD$ so the algebra is finitely generated ✓

So now why is the algebra $\mathfrak{R}$ on T saturated and semiample?

Let's review the definition of $\mathfrak{R}$.

Theorem 6.2

(X, Δ) log pair dim n

$f: X \to Z$ Z affine normal

k any $\mathbb{Z}_{>0}$ s.t. $k(K_X + \Delta)$ Cartier

Standard PLT assumptions

$\exists$ Birational $T \to S$

and $\forall\, m > 0$ $\exists$ log resolution $g_m : Y_m \to X$ s.t.

writing $K_{Y'_m} + \Gamma'_m =$

$$g_m^*(K_X + \Delta) + E_m$$

Γ'_m and E are effective w/ no common components

and letting

- N_m mobile part of $mK(K_{Y'_m} + \Gamma'_m)$
- $mK(K_{Y_m} + \Gamma_m)$ is log mobile part of $mK(K_{Y_m} + \Gamma'_m)$ (basically mobile part but get rid of negative coefficient part so Γ_m is still effective)
- $Z_m := E_m - \Gamma'_m + T$
- $\Theta_m := (\Gamma_m - T)|_T$
- M_m mobile part of $mK(K_T + \Theta_m)$

Then

Then

(i) Strict transform of S under all g_m is T and $Z_m|_T$ is independent of m ($Z_m|_T$ will be called Z, will be divisor F in definition of saturated)

There are common resolutions

$$\begin{tikzcd} Y_{m,n} \arrow[r] \arrow[d] & Y_m \arrow[d, "g_m"] \\ Y_n \arrow[r, "g_n"] & Y \end{tikzcd}$$

(ii) $N_m|_T = M_m$ and $|N_m|_T = |M_m|$

(iii) $\Theta_\bullet$ is convex (or at least $\Theta_{m\bullet}$ is for some $m \in \mathbb{Z}_{>0}$)

(iv) $\Theta_\bullet$ defines limiting algebra $\mathcal{R}$. $\mathcal{R}$ is fin gen iff $\mathcal{R}_S$ is.

Section 7 in the paper

Section 7 in the paper is there to show R is semiample.

Theorem 7.1. The base point free theorem holds for $\mathbb{R}$-divisors:

(X, Δ) $\mathbb{Q}$-factorial klt,
Δ - $\mathbb{R}$-divisor
D is nef $\mathbb{R}$-divisor s.t. $aD - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{R}_{>0}$
Then D is semiample.

This is the main tool we use to show things are semiample.

Extremely rough sketch

Extremely rough sketch of semiampleness:

Use MMP to construct models (finitely many) $T \overset{\psi_i}{\dashrightarrow} W_i$

s.t. $\forall K \gg 0$, $\exists i$ s.t.

$K_{W_i} + \psi_{i*}\Theta_K$ is semiample. (nef and use bpf theorem)

Construct common resolution

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ T \arrow[rr, dashed, "\psi_i"] & & W_i \end{tikzcd}$$

and use negativity of contraction, other birational stuff, to show the limit of characteristic sequence is semiample.

Saturation

Notation from theorem 6.2. M_m is mobile part of $mk(K_T+\Theta_m)$ (on T)

We might have to pass to a further model $T' \to T$ but we can assume $\exists s \in \mathbb{Z}_{>0}$ s.t. $P_m := \mathrm{Mob}(m\ell(K_T+\Theta_m))$ is free, $\ell := ks$

Q_m is mobile part of $m\ell(K_Y+\Gamma_m)$ (on T_m)

can assume Q_m is free

Note $sM_m \leq P_m \leq \underline{M_{ms}}$

Note $s\Gamma_m \le \Gamma_m \le \underline{M_{ms}}$

i.e.

$$sMob(mK(K_T + \Theta_m)) \le Mob(m\ell(K_T + \Theta_m)) \le Mob(msK(K_T + \Theta_{ms})) = M_{ms}$$

First bc $sMob(D) \le Mob(sD)$ for any D

Second bc $\Theta_m \le \Theta_{ms}$ consequence of convexity property.

Need to name the characteristic sequences

	on T		on Y_m
Mobile	$M_\bullet$	$P_\bullet$	$Q_\bullet$
Characteristic	$D_\bullet$	$I_\bullet$	$J_\bullet$

Lemma 9.1. $\forall i, j \in \mathbb{Z}_{>0}$

$$Mob(\lceil j J_i + Z_i \rceil)\big|_T \le M_{js}$$

Proof: Work on $Y_{i,js}$ the common resolution, so all divisors are pulled back to there are pulled back to there by appropriate maps.

Reminder: $J_i := \dfrac{\mathrm{Mob}(i\ell(K_i+\Gamma_i))}{i} \leq \ell(K_Y+\Gamma_i)$

so

$$\mathrm{Mob}(\lceil j J_i + Z_i\rceil) \leq \mathrm{Mob}(\lceil j\ell(K_{Y_i}+\Gamma_i)+Z_i\rceil)$$

$$\leq \mathrm{Mob}(\lceil j\ell(K_{Y_i}+\Gamma_i') + Z_i\rceil)$$

$$= \mathrm{Mob}(j\ell(K_{Y_i}+\Gamma_i') + \lceil Z_i\rceil)$$

$$= \mathrm{Mob}(j\ell g_i^*(K_X+\Delta) + j\ell E_i + \lceil Z_i\rceil)$$

because $j\ell(K_{Y_i}+\Gamma_i') = j\ell(g_i^*(K_X+\Delta)+E_i)$

bc $j\ell E_i + \lceil Z_i\rceil$ is effective and g_i-exceptional

$$= \mathrm{Mob}(j\ell g_j^*(K_X+\Delta))$$

$$\leq \mathrm{Mob}(j\ell g_{js}^*(K_X+\Delta) + j\ell E_{js})$$

$$= \mathrm{Mob}\big((js)K(K_{Y_{js}} + \Gamma'_{js})\big) = N_{js}$$

Now from 6.2 we know $N_{js}|_T = M_{js}$ so we get

$$\mathrm{Mob}(\lceil jJ_i + Z_i \rceil)|_T \leq M_{js}$$

as desired $\square$

Lemma 9.2.

The natural restriction map $H^0(Y_i, \mathcal{O}_{Y_i}(\lceil jJ_i + Z_i \rceil)) \longrightarrow H^0(T, \mathcal{O}_T(\lceil jI_i + Z \rceil))$

$$J_i|_T = \frac{Q_i}{i}\Big|_T = \frac{\mathrm{Mob}(ml(K_Y + \Gamma_m))}{i}\Big|_T$$

$$\overset{\text{by } 6.2?}{=} \underline{\mathrm{Mob}(m\ell(K_{T_i} + \theta_m))}_i = I_i ?$$

Restriction exact sequence

$$0 \to H^0(\lceil jJ_i + Z_i \rceil - T) \to H^0(\lceil jJ_i + Z_i \rceil) \to H^0(\lceil jI_i + Z \rceil) \to 0$$

surjective if

$$H^1(Y_i, \mathcal{O}_{Y_i}(\lceil jJ_i + (Z_i - T) \rceil))$$ expanding definition of Z_i

$$= H^1(Y_i, \mathcal{O}_{Y_i}(K_{Y_i} + \lceil g_i^*(-(K_X + \Delta) + jJ_i) \rceil))$$

Kawamata-Viehweg applies directly because $g^*(-(K_X + \Delta))$ is big and nef $(/Z)$ and jJ_i is nef (bc jJ_i is free)

Lemma 9.3. Suppose

Lemma 1.7. Suppose $-(K_X+\Delta)$ is nef and big. Then the s-truncation of the algebra $\mathfrak{R}$ is saturated w.r.t. Z.

We know $\lceil Z \rceil \geq 0$

and also

$$\mathrm{Mob}(\lceil j I_i + Z \rceil) \leq \mathrm{Mob}(\lceil j J_i + Z_i \rceil)\big|_T \leq M_{js}$$

$$\boxed{D_i = \frac{M_i}{i}}$$

thus

$$\mathrm{Mob}(\lceil j s D_{is} + Z \rceil) \leq \mathrm{Mob}(\lceil j I_i + Z \rceil)$$

$$\leq M_{js} \overset{\checkmark}{=} j s D_{is}$$

$$\leq M_{js} = jsD_{js}$$

first inequality bc

$$jsD_j = \frac{jsM_j}{j} \leq \frac{jP_i}{i} = jI_i.$$

For the truncation define $M'_j = M_{js}$, $D'_j = \frac{M'_j}{j} = sD_{js}$

so this gives

$$\mathrm{Mob}(\lceil jD'_j + Z \rceil) \leq \mathrm{Mob}(\lceil jI_j + Z \rceil)$$

$$\leq M_{js} = jD'_j$$

This is the saturation property we wanted $\square$

The only last subtle

The only thing is that chapter 7 proves $I_\circ$ has a semiample limit

but the inequalities $sD_i \leq I_i \leq sD_i$ is say $\lim I_\circ = s \lim D_\circ$ so no problem.